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On Networks of Non-Deterministic Automata

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In the present paper it is shown that in the structure theory of non-deterministic automata (NDA) it is sufficient to consider only two standard network forms. The conditions are stated under which an NDA can be isomorphically embedded in a network of smaller NDA's with proper output. It turns out that every finite NDA has a decomposition into a network of this type. Finally, the conditions for the existence of a decomposition the components of which realize the network output are derived. The results are stated and proved for the special case of two-component networks.

0. INTRODUCTION

In this paper we investigate the properties of finite non-deterministic (i.e. possibilistic) automata (ND-automata or NDA, for short), which are isomorphically embedded in networks of simpler NDA's. We start with the definition of different network conceptions and show that in structural investigations we can restrict our attention to two standard forms only.

The algebraic structure theory of non-deterministic automata is a generalization of the structure theory of finite deterministic automata [1], and it is strongly related to the problem of realizing a given NDA by a non-deterministic switching network, i.e. to the state coding problem for ND-automata. In [3] these questions had been investigated for a network conception the components of which are ND-semiautomata depending merely on the *present* states of the rest of the components in addition to the external input of the total network and their own *present* states. Hence, the next states of each component, which are members of a non-empty *set* of possible next states, must be chosen *independently* of the *next* states of all other components. This is an essential restriction leading to the fact that not every NDA can be embedded isomorphically into some network of this type. However, for every NDA $\mathscr{A}$ there exists a network of independent operating components embedding $\mathscr{A}$ *homomorphically* [3].

In the following, we investigate the effects of allowing the network to contain components with a nontrivial non-deterministic output. In this case, it turns out that *every* NDA is isomorphic to a subautomaton of a nontrivial network of this type.

A familiarity with [1] will facilitate the insight into this study.

1. BASIC DEFINITIONS

In the sequel we use the notation of [2].

1.1. Definition. The quadruple $\mathscr{A} = [X, Y, Z, h]$ is a *non-deterministic (synchronous) automaton* (NDA) provided that

(i) X, Y and Z are non-empty sets and

(ii) h maps $Z \times X$ uniquely into the set $\mathbf{P}^*(Y \times Z)$ of all non-empty subsets of $Y \times Z$*.

X, Y and Z, respectively, are called the input set, the output set and the set of the inner states of $\mathscr{A}$. In any case, Z is supposed to be finite. In every timing interval, $\mathscr{A}$ is in a certain state z in Z, reads some input signal x in X, and has the *possibility* to emit the signal y in Y and to go into state z' if and only if (iff) $[y, z'] \in h(z, x)$. Hence, in general, the output signal and the next state depend on one another. h can be decomposed in two ways:

If we define the functions g, f, h_y and h_z by

$$g(z, x) =_{\text{df}} \{y \mid \exists z'([y, z'] \in h(z, x))\},$$

$$h_y(z, x) =_{\text{df}} \{z' \mid [y, z'] \in h(z, x)\},$$

$$f(z, x) =_{\text{df}} \{z' \mid \exists y([y, z'] \in h(z, x))\},$$

$$h_{z'}(z, x) =_{\text{df}} \{y \mid [y, z'] \in h(z, x)\},$$

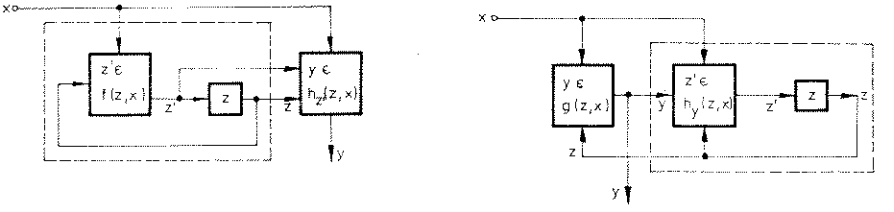
then we have

$$h(z, x) = \bigcup_{y \in g(z,x)} \{y\} \times h_y(z, x) = \bigcup_{z' \in f(z,x)} h_{z'}(z, x) \times \{z'\}$$

for all z, z' in Z, x in X and y in Y. $h_{z'}$ is called the conditional output function, h_y is the conditional next state function of $\mathscr{A}$, while g and f are called the output function and next state function, respectively.

* We use the symbol $\mathbf{P}(S)$ to denote the set of all subsets of the set S while the asterisk means that the empty subset is omitted.

From the above decomposition of h it is clear that we have two ways to represent an NDA by a "network" of output block and next state block (see Fig. 1). In structural observations we can use both modes. But since such observations are mainly concerned with the next state block, in Fig. 1(b) we can regard the output signal as an additional input signal for the state block. Therefore we can restrict ourselves to ND-*semiautomata*, denoted by $\mathscr{A} = [X, Z, f]$. After decomposing $\mathscr{A}$ into a network, as to be shown in the present paper, we can complete the realization by adding the output block as shown in Fig. 1. However, in section 4 we consider networks the output functions of which are realizable by their components alone.



(a) (b)

Fig. 1. Different modes of representing a NDA; (a) The output depends on the next state; (b) The next state depends on the present output.

1.2. Definition. For the given NDA's $\mathscr{A} = [X, Y, Z, h]$ and $\mathscr{A}' = [X', Y', Z', h']$, let ζ: $Z \to Z'$ be a one-to-one mapping. Then call ζ *Z-isomorphism* from $\mathscr{A}$ onto $\mathscr{A}'$ iff

$$[y, z'] \in h(z, x) \leftrightarrow [y, \zeta(z')] \in h'(\zeta(z), x)$$

for all x in X, y in Y and z, z' in Z. $\mathscr{A}$ is called Z-isomorphic to $\mathscr{A}'$ iff there exists a Z-isomorphism from $\mathscr{A}$ to $\mathscr{A}'$. For convenience, we shall denote ζ simply as isomorphism.

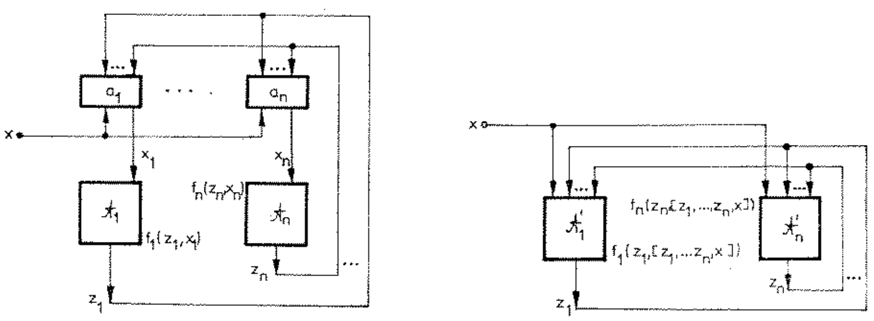
Clearly, the two semiautomata $\mathscr{A} = [X, Z, f]$ and $\mathscr{A}' = [X, Z', f']$ are isomorphic iff $z' \in f(z, x) \leftrightarrow \zeta(z') \in f'(\zeta(z), x)$.

2. CONCEPTIONS FOR NON-DETERMINISTIC AUTOMATA NETWORKS

The first part of this section is devoted to networks the components of which are ND-semiautomata. Of course, all automata networks are supposed to have at least two components.

2.1. Definition. $\mathscr{A} = [X, Z, f]$ is an ND-*network of the first kind* consisting of the components $\mathscr{A}_i = [X_i, Z_i, f_i]$ $(i = 1, \ldots, n;\ n \geqq 2)$ iff $Z = \bigtimes_{i=1}^{n} Z_i$ and there exist functions $a_i\colon \mathop{\times}\limits_{i=1}^{n} Z_i \times X \to \mathbf{P}^*(X_i)$ fulfilling $f([z_1, \ldots, z_n], x) = \mathop{\times}\limits_{i=1}^{n} f_i(z_i, a_i(z_1, \ldots, z_n, x))$ – see Fig. 2(a). According to [1] we call $\mathscr{A}$ network in delay form if every f_i depends only on x_i, i.e. not explicitely on state z_i. $\mathscr{A}$ is said to be in *standard form* iff there exist sets S_{ij}, S_j^X $(i, j = 1, \ldots, n)$ and functions $c_{ij}\colon Z_i \to S_{ij}$ and $c_j^X\colon X \to S_j^X$ such that $a_j(z_1, \ldots, z_n, x) = [c_{1j}(z_1), \ldots, c_{nj}(z_n), c_j^X(x)]$.

From 2.1 we see that, in general, $\mathscr{A}_i$ depends on its own state z_i in two ways: on the one hand the direct internal dependence on the state occupied by $\mathscr{A}_i$, and on the other hand the external way by function a_i (see [1, p. 82]). This is not the case if the network is in delay form. (However, we shall use networks which are not necessarily in delay form.) Without loss of generality, we regard only networks in standard form, the functions c_{ij} and c_j^X of which are the respective identities (see Fig. 2(b)). We denote a network $\mathscr{A}$ according to definition 2.1 by $N_1(\mathscr{A}_1, \ldots, \mathscr{A}_n, \{a_{i;i=1,\ldots,n}\})$ and a network in standard form by $N_1(\mathscr{A}_1, \ldots, \mathscr{A}_n)$.



(a) (b)

Fig. 2. First kind networks; (a) The network $\mathscr{A} = N_1(\mathscr{A}_1, \ldots, \mathscr{A}_n, \{a_{i,i=1,\ldots,n}\})$ consisting of ND-semiautomata; (b) The network $\mathscr{A} = N_1(\mathscr{A}_1', \ldots, \mathscr{A}_n')$ in standard form.

2.2. Proposition. For every network $\mathscr{A} = [X, Z, f] = N_1(\mathscr{A}_1, \ldots, \mathscr{A}_n, \{a_{i;i=1,\ldots,n}\})$ with components $\mathscr{A}_i = [X_i, Z_i, f_i]$, $i = 1, \ldots, n$, there exists an isomorphic network of the first kind in standard form $\mathscr{A}' = [X, Z, f'] = N_1(\mathscr{A}_1', \ldots, \mathscr{A}_n')$ where $\mathscr{A}_i' = [X_i', Z_i, f_i']$, $i = 1, \ldots, n$.

Proof. Combine the functions a_i and f_i to define f_i':

$$f_i'(z_i, [z_1, \ldots, z_n, x]) =_{\mathrm{df}} f_i(z_i, a_i(z_1, \ldots, z_n, x)) .$$

Obviously, $\mathscr{A}$ is isomorphic to $\mathscr{A}'$. □

In the structure theory, much effort is devoted to the problem of realizing a given automaton by a network having a reduced number of connections between its components. The loop-free network is one important network form with reduced complexity. The following definition gives a precise notion of loop-freedom.

2.3. Definition. The network $\mathscr{A}$ of the first kind with components $\mathscr{A}_1, \ldots, \mathscr{A}_n$ is called *loop-free* iff $f_i(z_i, [z_1, \ldots, z_i, z_{i+1}, \ldots, z_n, x]) = f_i(z_i, [z_1, \ldots, z_i, z'_{i+1}, \ldots, z'_n, x])$ for all $i = 1, \ldots, n$, z_i, z'_i in Z_i and x in X.

In the present paper, by virtue of 2.2 it is sufficient to restrict our attention to networks in standard form.

2.4. Definition. $\mathscr{A} = [X, Z, f]$ is a *network of type 2A* consisting of $\mathscr{A}_i = [X_i, Z_i, f_i]$, $i = 1, \ldots, n$, provided there exists a function

$$c: \mathop{\times}_{i=1}^{n} Z_i \times X \to \mathbf{P}^*(\mathop{\times}_{i=1}^{n} X_i)$$

where

$$[z'_1, \ldots, z'_n] \in f([z_1, \ldots, z_n], x) \leftrightarrow$$

$$\leftrightarrow \exists x_1 \, \exists x_2 \ldots \exists x_n([x_1, \ldots, x_n] \in c(z_1, \ldots, z_n, x) \wedge \bigwedge_{i=1}^{n} z'_i \in f_i(z_i, x_i))$$

holds. We denote such networks by $N_{2A}(\mathscr{A}_1, \ldots, \mathscr{A}_n, c)$.

More generally, we regard networks the components of which do not only emit their present states but have in addition proper non-deterministic outputs. Hence, the output sets have the form $Y'_i = Z_i \times Y_i$. However, we will treat only the Y_i's as the proper output sets.

It is evident that no component is allowed the input signal of which depends on its own output signal at the same clock period.

Now we define this network conception, and then we show that this one and the conception of 2.4 are both equivalent to the same standard form. This allowes us to use standard forms only.

2.5. Definition. $\mathscr{A} = [X, Z, f]$ is a *network of type 2B* consisting of the components $\mathscr{A}_i = [X_i, Y_i, Z_i, h_i]$, $i = 1, \ldots, n$, iff $Z = \mathop{\times}_{i=1}^{n} Z_i$ and there exist functions

$$c_i: \mathop{\times}_{j=1}^{i-1} Y_j \times \mathop{\times}_{j=i}^{n} Z_j \times X \to \mathbf{P}(X_i), \quad i = 1, \ldots, n,$$

such that

$$[z'_1, \ldots, z'_n] \in f([z_1, \ldots z_n], x) \leftrightarrow$$

$$\leftrightarrow \exists x_1 \, \exists x_2 \ldots \exists x_n \, \exists y_1 \, \exists y_2 \ldots \exists y_n \, \forall i(\bigwedge_{j=1}^{n} x_j \in X_j \wedge \bigwedge_{j=1}^{n} y_j \in Y_j \wedge$$

$$\wedge \; i \in \{1, \ldots, n\} \to x_i \in c_i(y_1, \ldots, y_{i-1}, z_i, \ldots, z_n, x) \wedge$$

$$\wedge \; [y_i, z'_i] \in h_i(z_i, x_i)) .$$

2.6. Definition. A network of type 2B is in second kind *standard form* iff the values of all c_i's are either singletons or empty.

Without loss of generality, we can identify all X_i's with a subset of $\bigtimes_{j=1}^{i-1} Y_j \times \bigtimes_{j=1}^{n} Z_j \times X$ for $i \in \{1, \ldots, n\}$. Hence, we can regard the c_i's to be partially defined mappings onto the X_i's.

Networks according to 2.5 and 2.6, respectively, are denoted by $N_{2B}(\mathscr{A}_1, \ldots, \mathscr{A}_n, \{c_{i;i=1,\ldots,n}\})$ and $N_2(\mathscr{A}_1, \ldots, \mathscr{A}_n)$.

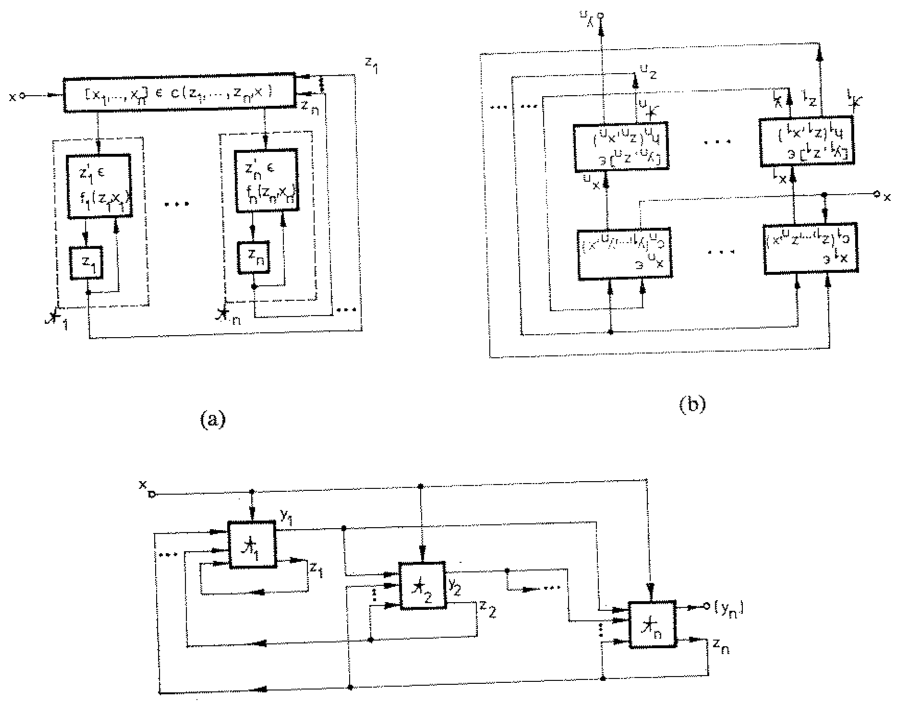


Fig. 3. Second kind networks; (a) Type 2A; (b) Type 2B; (c) Second kind standard form.

2.7. Proposition. For every network $\mathscr{A} = N_{2B}(\mathscr{A}_1, \ldots, \mathscr{A}_n, \{c_{i;i=1,\ldots,n}\})$ where $\mathscr{A}_i = [X_i, Y_i, Z_i, h_i]$ $(i = 1, \ldots, n)$ there exists an isomorphic network $\mathscr{A}' = N_2(\mathscr{A}'_1, \ldots, \mathscr{A}'_n)$ in standard form consisting of $\mathscr{A}'_i = [X'_i, Y_i, Z_i, h'_i]$ for $i = 1, \ldots, n$.

Proof. Define for $i = 1, \ldots, n$ the input sets $X'_i = \bigtimes_{j=1}^{i-1} Y_j \times \bigtimes_{j=i}^{n} Z_j \times X$ and $h'_i(z_i, x'_i) =_{df} h_i(z_i, c_i(x'_i))$ for all z_i in Z_i and for those x'_i in X'_i such that $c_i(x'_i) \neq \emptyset$. Don't care conditions result for all other x'_i. It can be easily shown that $\mathscr{A}'$ is isomorphic to $\mathscr{A}$. □

2.8. Proposition. For every network $\mathscr{A} = N_{2A}(\mathscr{A}_1, \ldots, \mathscr{A}_n, c)$ where $\mathscr{A}_i = [X_i, Z_i, f_i]$ $(i = 1, \ldots, n)$ there exists an isomorphic network $N_2(\mathscr{A}'_1, \ldots, \mathscr{A}'_n)$ in standard form consisting of $\mathscr{A}'_i = [X'_i, Y_i, Z_i, h'_i]$ $(i = 1, \ldots, n)$.

Proof. Define $Y_i = X_i$ and $X'_i = \bigtimes_{j=1}^{i-1} Y_j \times \bigtimes_{j=i}^{n} Z_j \times X$ and for all $i = 1, \ldots, n$:

$$[x_i, z'_i] \in h'_i(z_i, [y_1, \ldots, y_{i-1}, z_i, \ldots, z_n, x]) \leftrightarrow_{\mathrm{df}}$$
$$\leftrightarrow_{\mathrm{df}} \exists x_{i+1} \ldots \exists x_n([y_1, \ldots, y_{i-1}, x_i, x_{i+1}, \ldots, x_n] \in$$
$$\in c(z_1, \ldots, z_n, x) \wedge z'_i \in f_i(z_i, x_i)).$$

This implies

$$[z'_1, \ldots, z'_n] \in f([z_1, \ldots, z_n], x) \leftrightarrow_{2.5}$$
$$\leftrightarrow_{2.5} \exists x_1 \ldots \exists x_n([x_1, \ldots, x_n] \in c(z_1, \ldots, z_n, x) \wedge \bigwedge_{i=1}^{n} z'_i \in f_i(z_i, x_i)) \leftrightarrow$$
$$\leftrightarrow [z'_1, \ldots, z'_n] \in f'([z_1, \ldots, z_n], x)$$

which was to be proved. □

2.9. Definition. The second kind network $\mathscr{A} = N_2(\mathscr{A}_1, \ldots, \mathscr{A}_n)$ is called *loop-free* provided there exists a permutation p of $\{1, \ldots, n\}$ such that for $i, j \in \{1, \ldots, n\}$,

$$X_{i_j} = \begin{cases} Y_j & \text{if } j < i, \\ Z_j & \text{otherwise} \end{cases} \quad \text{and } x_{i_j}, x'_{i_j} \text{ in } X_{i_j}, z_i \text{ in } Z_i:$$

$$X_i = \bigtimes_{j=1}^{n} X_{i_{p(j)}} \quad \text{and}$$

$$f_{p(i)}(z_{p(i)}, [x_{p(i)_{p(1)}}, \ldots, x_{p(i)_{p(i)}}, x_{p(i)_{p(i+1)}}, \ldots, x_{p(i)_{p(n)}}, x]) =$$
$$= f_{p(i)}(z_{p(i)}, [x_{p(i)_{p(1)}}, \ldots, x_{p(i)_{p(i)}}, x'_{p(i)_{p(i+1)}}, \ldots, x'_{p(i)_{p(n)}}, x]).$$

In other words, the $p(i)$-th component depends only on the components $p(1), \ldots, p(i)$ and on x.

3. ISOMORPHIC EMBEDDING OF NON-DETERMINISTIC AUTOMATA INTO NETWORKS

An NDA $\mathscr{A}$ is called isomorphically *embedded* in an NDA $\mathscr{A}''$ iff $\mathscr{A}$ is isomorphic to a subautomaton $\mathscr{A}'$ of $\mathscr{A}''$. Let $\mathscr{A}'' = [X, Z'', f'']$ be a network (of the first or second kind) which $\mathscr{A}$ is isomorphically embedded in. Each component $\mathscr{A}_i$ of $\mathscr{A}''$ generates a *partition* τ_i of the state set Z of $\mathscr{A}$ by means of the isomorphism ζ from the subautomaton $\mathscr{A}' = [X, Z', f']$ of $\mathscr{A}''$ onto $\mathscr{A}$:

$$\tau_i = \{\{z \mid \zeta^{-1}(z) \in Z_1 \times \ldots \times Z_{i-1} \times \{z_i\} \times Z_{i+1} \times \ldots \times Z_n\} \mid z_i \in Z_i\}.$$

(Thereby, empty sets are omitted.) Define the *product* of two partitions to be the set of all non-empty intersections of their elements (blocks). Then the product $\prod_{i=1}^{n} \tau_i$ of all partitions τ_i equals the *zeropartition* 0, i.e. the partition containing singletons only. Note that some of the τ_i's could be 1-partitions, i.e. the trivial partition $\{Z\}$. However, throughout this paper we use only nontrivial embeddings, i.e. (i) each component has fewer states than $\mathscr{A}$ and (ii) for each component $\mathscr{A}_i$ there exists a state $z = [z_1, \ldots, z_i, \ldots, z_n]$ of $\mathscr{A}'$ such that $\mathscr{A}_i$ can leave z_i whenever $\mathscr{A}'$ leaves z.

Let ζ_i project the inverse ζ^{-1} of ζ onto the i-th component Z_i of Z''. Then $\zeta_i(z) = \zeta_i(z')$ for all $z, z' \in N \in \tau_i$. Therefore we call $\zeta_i(N)$ the image of *some* z in N under ζ_i. A partition π is *not greater* then τ ($\pi \leqq \tau$) iff every π-block is contained in some τ-block. Then $\pi < \tau$ iff $\pi \leqq \tau$ and $\pi \neq \tau$.

3.1. Networks of the first kind

This network conception and the corresponding decomposition theory was studied in [3]. In this section we shall state the main results briefly.

3.1. Definition. Let $\mathbf{M} = \{\tau_1, \ldots, \tau_n\}_{n>0}$ be a set of partitions of the state set Z of $\mathscr{A} = [X, Z, f]$. $\mathbf{M}$ is called *independent* (with respect to $\mathscr{A}$) provided that for all z in Z, x in X and N_i in τ_i, $i = 1, \ldots, n$, the condition

$$\bigwedge_{i=1}^{n} (N_i \cap f(z, x) \neq \emptyset) \leftrightarrow \bigcap_{i=1}^{n} N_i \cap f(z, x) \neq \emptyset$$

holds.

The reader will easily see that for given $\mathscr{A}$, in general, not every $\mathbf{M}$ is independent.

The following theorem answers the question whether or not a given NDA is decomposable into a network of the first kind.

3.2. Theorem. An NDA $\mathscr{A}$ can be isomorphically embedded into some network of the first kind consisting of n component automata iff there exists an independent set $\mathbf{M}$ of n partitions of Z where $\prod_{\tau \in \mathbf{M}} \tau = 0$ and $0 < \tau < 1$ for all $\tau \in \mathbf{M}$.

Proof. Define $\mathscr{A}_1, \ldots, \mathscr{A}_n$ by $\mathscr{A}_i =_{\mathrm{df}} [X_i, Z_i, f_i]$, $i = 1, \ldots, n$, where

$$X_i = \tau_1 \times \ldots \times \tau_{i-1} \times \tau_{i+1} \times \ldots \times \tau_n \times X, \quad Z_i = \tau_i$$

and

$$f_i(N_i, [N_1, \ldots, N_{i-1}, N_{i+1}, \ldots, N_n, x]) = \begin{cases} f(\iota z(z \in \bigcap_{j=1}^{n} N_j), x), & \text{iff } \bigcap_{j=1}^{n} N_j \neq \emptyset; \\ \text{“don't care condition”, otherwise.*} \end{cases}$$

It can be shown [3] that $\mathscr{A}$ is isomorphically embedded in $N_1(\mathscr{A}_1, \ldots, \mathscr{A}_n)$. □

* $\iota z(P(z))$ denotes the *unique* z fulfilling P.

Next we define the conceptions of partition pair and substitution property partition for NDA's which are important when reduced dependences between the components are regarded.

3.3. Definition. $[\pi, \tau]$ is a *partition pair* (PP) of $\mathscr{A}$ iff for all x in X, N in π, z, z' in N and N' in τ,

$$N' \cap f(z, x) \neq \emptyset \leftrightarrow N' \cap f(z', x) \neq \emptyset .$$

τ is a *substitution property partition* (SP-partition or SPP, for short) of $\mathscr{A}$ iff $[\tau, \tau]$ is a PP.

3.4. Theorem. An NDA $\mathscr{A}$ can be isomorphically embedded into a loop-free network of the first kind consisting of n components iff there exist a set $\mathbf{M}$ of SPP's $\pi_1, \ldots, \pi_n$, a set $\mathbf{N}$ of partitions $\tau_1, \ldots, \tau_m$ and a one-to-one mapping F from $\mathbf{M}$ onto $\mathbf{N}^\dagger =_{\mathrm{df}} \max(\mathbf{M}) \cup \mathbf{N}$, where

(i) $F(\pi) = \pi$ for π in $\max(\mathbf{M})$;

(ii) $F(\pi) = \tau > \pi$ for π in $\mathbf{M} \setminus \max(\mathbf{M})$ and some τ in $\mathbf{N}$;

(iii) $\prod_{\substack{\pi' \in \mathbf{M} \\ \pi' > \pi}} \pi' \,.\, F(\pi) \leqq \pi$;

(iv) $\prod_{\tau \in \mathbf{N}^\dagger} \tau = 0$ and $\prod_{\pi \in \mathbf{M}} \pi = 0$;

(v) $\mathbf{N}^\dagger$ is independent.

If $\max(\mathbf{M}) = \mathbf{M}$ then each component will operate independently of all other components. (Such a network sometimes is refered to as "parallel composition".)

Proof (see [3]). Construct the network $N_1(\mathscr{A}_1, \ldots, \mathscr{A}_n)$ similarly to that of Theorem 3.2 using the partitions in $\mathbf{N}^\dagger$. However, one can show that because of (iii) the component $\mathscr{A}_i$, $i < n$, will not depend on component $\mathscr{A}_j$, $j > i$. Hence $N_1(\mathscr{A}_1, \ldots, \mathscr{A}_n)$ is loop-free. □

3.2. Networks of the second kind

In this paper, we shall restrict our attention to networks of two components only, as it is usually done in literature. That will do for our purpose, i.e. to state the characteristic properties of an NDA that can be embedded into a network of the second kind. Hence, all statements can be easily generalized to networks of more than two components. That is, we regard networks $\mathscr{A}'' = [X, Z'', f'']$ consisting of the components $\mathscr{A}_1 = [X_1, Y_1, Z_1, h_1]$ and $\mathscr{A}_2 = [X_2, Z_2, f_2]$. Without loss of generality, we suppose $\mathscr{A}_2$ to be a semiautomaton for $\mathscr{A}_1$ is independent of the (proper) output of $\mathscr{A}_2$. Therefore, we have $X_1 = X \times Z_2, X_2 = X \times Y_1, Z'' =$

$= Z_1 \times Z_2$ and

$$[z_1', z_2'] \in f''([z_1, z_2], x) \leftrightarrow$$
$$\leftrightarrow z_1' \in f_1(z_1, [x, z_2]) \wedge z_2' \in f_2(z_2, h_{1;z'_1}(z_1, [x, z_2]))\,.$$

The following Lemma will be used when proving the main result 3.11.

3.5. Lemma. For all z_1 in Z_1, z_2 in Z_2 and x in X,

$$[z_1', z_2'] \in f''([z_1, z_2], x) \leftrightarrow$$
$$\leftrightarrow \exists y\, \forall \bar{z}_1\, \exists \bar{z}_1'(y \in h_{1;z'_1}(z_1, [x, z_2]) \wedge \bar{z}_1, \bar{z}_1' \in Z_1 \wedge$$
$$\wedge\ y \in g_1(\bar{z}_1, [x, z_2]) \wedge \bar{z}_1' \in h_y(\bar{z}_1, [x, z_2]) \wedge$$
$$\wedge\ [\bar{z}_1', z_2'] \in f''([\bar{z}_1, z_2], x))\,.$$

Proof. Let $[z_1', z_2'] \in f''([z_1, z_2], x)$. Then there exists some y in $h_{1;z'_1}(z_1, [x, z_2])$ such that $z_2' \in f_2(z_2, y)$. Supposed no such y fulfills the assertion of the theorem. Then for all $y \in h_{1;z'_1}(z_1, [x, z_2])$ there exists some $\bar{\bar{z}}_1 \in Z_1$ for which $y \in g_1(\bar{\bar{z}}_1, [x, z_2])$ such that for all $\bar{\bar{z}}_1' \in Z_1$, $\bar{\bar{z}}_1'$ is not in $h_y(\bar{\bar{z}}_1, [x, z_2])$ or $[\bar{\bar{z}}_1', z_2']$ is not in $f''([\bar{\bar{z}}_1, z_2], x)$, i.e. $h_y(\bar{\bar{z}}_1, [x, z_2])$ is empty or z_2' is not a next state of the second component for any y. This contradicts the suppositon. The rest of the proof is trivial. □

3.6. Definition. Let τ be a partition of Z. Define $f^\tau \subseteq Z \times X \times \tau$ by $[z, x, N] \in f^\tau \leftrightarrow \leftrightarrow f(z, x) \cap N \neq \emptyset$. f^τ is called the *transitional relation* of $\mathscr{A}$ with respect to τ. Then let $\mathbf{R}$ be a system of nonempty subsets of f^τ covering f^τ, i.e. $\bigcup \mathbf{R} = f^\tau$.

In 3.7 and 3.8, τ_1 and τ_2 are the respective partitions of Z generated by $\mathscr{A}_1$ and $\mathscr{A}_2$, where $\mathscr{A} = [X, Z, f]$ and $\mathscr{A}$ is isomorphic to the subautomaton $\mathscr{A}' = [X, Z', f']$ of $\mathscr{A}''$. ζ is the corresponding isomorphism from $\mathscr{A}'$ onto $\mathscr{A}$.

3.7. Definition. Define the unique functions u: $Z \times \mathbf{R} \times X \to \mathbf{P}(Z)$ and r: $Y_1 \to \to \mathbf{P}^*(Z \times X \times \tau_1)$, respectively, by

$$u(z, R, x) = \bigcup\{N \mid [z, x, N] \in R\}$$

and

$$r(y) = \{[z, x, N] \mid y \in h_{1;\zeta_1(N)}(\zeta_1(z), [x, \zeta_2(z)]),$$

where $z \in Z$, $x \in X$, $R \in \mathbf{R}$ and $y \in Y_1$. u is called the function associated with $\mathbf{R}$. $\mathbf{R}^{Y_1} =_{\mathrm{df}} \{r(y) \mid y \in Y_1\}$ covers f^{τ_1}, and r maps Y_1 onto $\mathbf{R}^{Y_1}$. $\mathbf{R}$ is called τ_2-*admissible* (with respect to $\mathscr{A}$), iff for all x in X

$$\forall N_2\, \forall R\, \exists N_2'\, \forall N_1(N_2, N_2' \in \tau_2 \wedge N_1 \in \tau_1 \wedge (N_1 \cap N_2) \times \{x\} \times \tau_1 \cap R \neq \emptyset \to$$
$$\to N_2' \cap u(\varepsilon z(z \in N_1 \cap N_2), R, x) \cap f(\varepsilon z(z \in N_1 \cap N_2), x) \neq \emptyset)\,.$$

Thereby the ε-operator denotes any element fulfilling the subsequent predicate. We shall use only admissible $\mathbf{R}$'s.

3.8. Proposition. For all $z \in Z$, $R \in \mathbf{R}^{Y_1}$ and $x \in X$:

$$u(z, R, x) = \zeta(h_{1_{r^{-1}(R)}}(\zeta_1(z), [x, \zeta_2(z)]) \times Z_2) .$$

Proof. From the definition of function r,

$$\zeta(h_{1_{r^{-1}(R)}}(\zeta_1(z), [x, \zeta_2(z)]) \times Z_2) = \zeta(h_{1_y}(\zeta_1(z), [x, \zeta_2(z)]) \times Z_2)$$

for some y such that $y \in h_{1;\zeta_1(N)}(\zeta_1(z), [x, \zeta_2(z)])$ for all $[z, x, N] \in R$. Therefore the above expression equals

$$\zeta(\{\zeta_1(N) \mid [z, x, N] \in R\} \times Z_2) = \bigcup\{N \mid [z, x, N] \in R\} = u(z, R, x) . \qquad \square$$

3.9. Definition. τ_2 **R**-*depends* on τ_1 iff for all x in X, N_1, N_1' in τ_1, N_2, N_2' in τ_2 such that $N_1 \cap N_2 \neq \emptyset$ and some z in $N_1 \cap N_2$ the following condition holds:

$$N_1' \cap N_2' \cap f(z, x) \neq \emptyset \leftrightarrow$$
$$\leftrightarrow \exists R(R \in \mathbf{R} \wedge [z, x, N_1'] \in R \wedge \forall \bar{z}(\bar{z} \in N_2 \wedge \{\bar{z}\} \times \{x\} \times \tau_1 \cap R \neq \emptyset \rightarrow$$
$$\rightarrow N_2' \cap u(\bar{z}, R, x) \cap f(\bar{z}, x) \neq \emptyset)) .$$

In the following, we use $a \equiv b(\mathbf{C})$ to denote the fact that if $\mathbf{C}$ is a set of subsets of the set S, then elements a and b are in the same subset.

3.10. Proposition. If $\tau_1 . \tau_2 = 0$ then $\{\tau_1, \tau_2\}$ is independent according to 3.1 iff there exists an **R** (for f^{τ_1}) such that for all z, z' in Z, x, x' in X and N, N' in τ_1 $[z, x, N] \not\equiv [z', x', N'] (\mathbf{R}) \rightarrow z \not\equiv z'(\tau_1)$ holds while τ_2 **R**-depends on τ_1.

Proof. Suppose $\{\tau_1, \tau_2\}$ is independent. Define **R** by $[z, x, N] \equiv [z', x', N'] (\mathbf{R}) \leftrightarrow_{\text{df}} z \equiv z'(\tau_1)$. We show that τ_2 **R**-depends on τ_1. To do this we suppose $N_1' \cap N_2' \cap f(z, x) \neq \emptyset$ for $z \in N_1 \cap N_2$, $N_i \in \tau_i$, $i = 1, 2$, and $x \in X$. This is true if and only if $N_1' \cap f(z, x) \neq \emptyset$ and $N_2' \cap f(z, x) \neq \emptyset$. Therefore, $[z, x, N_1']$ is in f^{τ_1}. There exists a unique R in **R** containing $[z, x, N_1']$. Show that this R fulfills the expression of 3.9. At first we note that from $\tau_1 . \tau_2 = 0$ and the way **R** is defined, z is the only $\bar{z}$ for which in 3.9 the conclusion must hold. Then, R contains all $[z, x, N_1'']$ such that $[z, x, N_1'']$ is in f^{τ_1}. Hence $u(z, R, x) \supseteq f(z, x)$ and the conclusion reduces to $N_2' \cap f(z, x) \neq \emptyset$. This proves the first part of the proposition. Similarly one proves the rest. $\square$

The following theorem is the main result of this paper.

3.11. Theorem. $\mathscr{A}$ is isomorphic to the subautomaton $\mathscr{A}'$ of the network $\mathscr{A}''$ of the second kind consisting of $\mathscr{A}_1 = [X_1, Y_1, Z_1, h_1]$ and $\mathscr{A}_2 = [X_2, Z_2, f_2]$ iff there exist nontrivial partitions τ_1 and τ_2 and an **R** for f^{τ_1} such that $\tau_1 . \tau_2 = 0$ and τ_2 **R**-depends on τ_1.

Proof. (1) Let $\mathscr{A}$ be isomorphic to $\mathscr{A}'$. Define $\mathbf{R} =_{\mathrm{df}} \{r(y) \mid y \in Y_1\}$. Suppose that for x in X, N_1, N'_1 in τ_1, N_2, N'_2 in τ_2, z_i in N_i and z'_i in N'_i $(i = 1, 2)$ and for some z in $N_1 \cap N_2 \neq \emptyset$

$$\exists R(R \in \mathbf{R} \wedge [z, x, N'_1] \in R \wedge \forall \bar{z}(\bar{z} \in N_2 \wedge \{\bar{z}\} \times \{x\} \times \tau_1 \cap R \neq \emptyset \to$$
$$\to N'_2 \cap u(\bar{z}, R, x) \cap f(\bar{z}, x) \neq \emptyset))$$

is true. Let $r^{-1}(R)$ be a fixed y in Y_1 such that $r(y) = R$. Then the above expression is equivalent to

$$\exists y(y \in Y_1 \wedge y \in h_{1;z'_1}(z_1, [x, z_2]) \wedge \forall \bar{z}_1(\bar{z}_1 \in Z_1 \wedge y \in g_1(\bar{z}_1, [x, z_2]) \to$$
$$\to Z_1 \times \{z'_2\} \cap h_{1_y}(\bar{z}_1, [x, z_2]) \times Z_2 \cap f'([\bar{z}_1, z_2], x) \neq \emptyset)) \Leftrightarrow$$
$$\Leftrightarrow \exists y(y \in Y_1 \wedge y \in h_{1;z'_1}(z_1, [x, z_2]) \wedge \forall \bar{z}_1(\bar{z}_1 \in Z_1 \wedge y \in g_1(\bar{z}_1, [x, \bar{z}_2)] \to$$
$$\to h_{1_y}(\bar{z}_1, [x, z_2]) \times \{z'_2\} \cap f'([\bar{z}_1, z_2], x) \neq \emptyset)) \Leftrightarrow$$
$$\Leftrightarrow \exists y(y \in h_{1;z'_1}(z_1, [x, z_2]) \wedge \forall \bar{z}_1 \exists \bar{z}'_1(\bar{z}_1 \in Z_1 \wedge y \in g_1(\bar{z}_1, [x, \bar{z}_2]) \wedge$$
$$\wedge \bar{z}'_1 \in h_{1_y}(\bar{z}_1, [x, z_2]) \wedge [\bar{z}'_1, z'_2] \in f'([\bar{z}_1, z_2], x))) \Leftrightarrow$$
$$\Leftrightarrow_{3.5} [z'_1, z'_2] \in f''([z_1, z_2], x) \Leftrightarrow N'_1 \cap N'_2 \cap f(z, x) \neq \emptyset .$$

It can be easily shown that $\mathbf{R}$ is a τ_2-admissible set system.

(2) For given $\mathscr{A}$ let $\tau_1 . \tau_2 = 0$ and $\mathbf{R}$ be a set system covering f^{τ_1} such that τ_2 $\mathbf{R}$-depends on τ_1. Then define

$$\mathscr{A}_1 = [X \times \tau_2, \mathbf{R}, \tau_1, h_1] ,$$
$$\mathscr{A}_2 = [X \times \mathbf{R}, \tau_2, f_2] ,$$
$$\mathscr{A}'' = [X, \tau_1 \times \tau_2, f''] ,$$
$$\mathscr{A}' = [X, Z', f']$$

where

$$Z' =_{\mathrm{df}} \{[N_1, N_2] \mid [N_1, N_2] \in Z'' \wedge N_1 \cap N_2 \neq \emptyset\} , \quad \zeta : Z' \to\to Z ,$$

where $\zeta([N_1, N_2]) = \iota z(z \in N_1 \cap N_2)$ and for $[N_1, N_2] \in Z'$:

$$[R, N'_1] \in h_1(N_1, [x, N_2]) \leftrightarrow_{\mathrm{df}} [z, x, N'_1] \in R$$

and

$$N'_2 \in f_2(N_2, [x, R]) \leftrightarrow_{\mathrm{df}} \forall \bar{N}_1(\bar{N}_1 \in \tau_1 \wedge (\bar{N}_1 \cap N_2) \times \{x\} \times \tau_1 \cap R \neq \emptyset \to$$
$$\to N'_2 \cap u(\iota \bar{z}(\bar{z} \in \bar{N}_1 \cap N_2), R, x) \cap f(\bar{N}_1 \cap N_2, x) \neq \emptyset) .$$

 Since $\mathbf{R}$ is τ_2-admissible, f_2 is not empty. We prove that ζ is an isomorphism. (Then $\mathscr{A}'$ is closed, i.e. $\mathscr{A}'$ is a subautomaton of $\mathscr{A}''$.)

Let $\{z\} = N_1 \cap N_2$, $\{z'\} = N_1' \cap N_2'$ and $x \in X$. Then we have

$$z' \in f(z, x) \leftrightarrow N_1' \cap N_2' \cap f(z, x) \neq \emptyset \leftrightarrow$$

$$\leftrightarrow \exists R(R \in \mathbf{R} \wedge [z, x, N_1'] \in R \wedge \forall \bar{z}(\bar{z} \in N_2 \wedge \{\bar{z}\} \times \{x\} \times \tau_1 \cap R \neq \emptyset \rightarrow$$

$$\rightarrow N_2' \cap u(\bar{z}, R, x) \cap f(\bar{z}, x) \neq \emptyset)) \leftrightarrow$$

$$\leftrightarrow_{(\text{df. of } \mathscr{A}_1 \& \mathscr{A}_2)} \exists R(R \in Y_1 \wedge R \in h_{1;N'_1}(N_1, [x, N_2]) \wedge N_2' \in f_2(N_2, [x, R])) \leftrightarrow$$

$$\leftrightarrow [N_1', N_2'] \in f''([N_1, N_2], x) \leftrightarrow$$

$$\leftrightarrow [N_1', N_2'] \in f'([N_1, N_2], x) \leftrightarrow$$

$$\leftrightarrow \zeta^{-1}(z') \in f'(\zeta^{-1}(z), x)$$

which was to be shown. □

If there exists an $\mathbf{R}$ according to 3.10 then $\mathscr{A}$ can be realized by a network the two components of which operate independently of one another in the sense that, to perform its own operation, no component needs information about the *next* state of the other one. A first kind network will result in this case. The component $\mathscr{A}_2$ of $\mathscr{A}$ depends to the highest degree on $\mathscr{A}_1$ iff there is no $\mathbf{R} > \mathbf{R}_0 =_{\text{df}} =_{\text{df}} \{\{[z, x, N_1] \mid x \in X \wedge [z, x, N_1] \in f^{\tau_1}\} \mid z \in Z \wedge N_1 \in \tau_1\}$ such that τ_2 $\mathbf{R}$-depends on τ_1. For every τ_1, any τ_2 $\mathbf{R}_0$-depends on τ_1. Hence we have:

3.12. Corollary. For every finite NDA $\mathscr{A}$ there exists an $\mathbf{R} \geq \mathbf{R}_0$ such that $\mathscr{A}$ can be embedded isomorphically into a network of the second kind where $\mathscr{A}_1 = [X \times \tau_2, \mathbf{R}, \tau_1, h_1]$. □

Investigating loop-free networks here we are concerned only with the case that component $\mathscr{A}_1$ does not depend on component $\mathscr{A}_2$, for the remaining case would lead to a first kind network treated in section 3.1.

3.13. Definition. τ is an $\mathbf{R}$-*SP-partition* (or has the *substitution property* with respect to $\mathbf{R}$) iff

$$[z, x, N'] \in R \leftrightarrow [z', x, N'] \in R$$

for all R in $\mathbf{R}$, z, z' in Z such that $z \equiv z'(\tau)$, N' in τ and x in X.

3.14. Proposition. (i) τ is an $\mathbf{R}$-SP-partition implies that τ is an SP-partition. (ii) Let τ be a partition and let $\mathbf{R}^*$ be any set of subsets of f^τ which covers f^τ such that $z \not\equiv z'(\tau)$ implies $[z, x, N] \not\equiv [z', x', N']\ (\mathbf{R}^*)$. Then τ is an $\mathbf{R}^*$-SP-partition if τ is an SP-partition.

Proof. The proposition follows directly from the definitions of SP-partition, **R**-SP-partition and that of $\mathbf{R}^*$. □

3.15. Theorem. The NDA $\mathscr{A}$ can be embedded into a network $\mathscr{A}''$ of the second kind (with two components) the first component of which operates independently of the second one, iff there exist partitions τ_1 and τ_2 and an **R** for τ_1 such that

(i) τ_1 is an **R**-SP-partition;

(ii) $\tau_1 . \tau_2 = 0$;

(iii) τ_2 **R**-depends on τ_1.

Proof. Similar to that of 3.11. Since τ_1 is an **R**-SP-partition, h_1 does not depend on τ_2. □

4. THE REALIZATION OF THE NETWORK OUTPUT BY COMPONENTS

In the foregoing sections, we investigated the "decomposition properties" of ND-semiautomata. From that we saw that the "state behaviour" of the given NDA $\mathscr{A}$ can be realized by (decomposed into) a network which operates up to a one-to-one correspondence in the same manner as the state block of $\mathscr{A}$ (see Fig. 1 (a) or (b)). Thus, $\mathscr{A}$ can be completely realized by simply adding the output block to the network. Now we shall study the properties of an NDA $\mathscr{A}$ which is "completely decomposable" that is, it can be decomposed into a network of smaller NDA's which, *in common*, realize the output of $\mathscr{A}$, too.

4.1. Definition. Let $\mathscr{A} = [X, Y, Z, h]$ be a network of the first or second kind, respectively, consisting of $\mathscr{A}_1, \ldots, \mathscr{A}_n$. The components *realize the network output* iff $\mathscr{A}_i = [X_i, Y_i, Z_i, h_i]$, $i = 1, \ldots, n$, $Y = \bigtimes_{i=1}^{n} Y_i$ and $[y, z'] \in h(z, x) \leftrightarrow \bigwedge_{i=1}^{n} [y_i, z'_i] \in h_i(z_i, x_i)$ where $y = [y_1, \ldots, y_n]$, $z = [z_1, \ldots, z_n]$, $z' = [z'_1, \ldots, z'_n]$ and $x_i = [z_1, \ldots, z_n, x]$ and $x_i = [y_1, \ldots, y_{i-1}, z_i, \ldots, z_n, x]$, respectively.

4.2. Definition. Let ϱ_i be partitions of Y, τ_i partitions of Z and $\chi_i = [\varrho_i, \tau_i]$, $i = 1, \ldots, n$, for $\mathscr{A} = [X, Y, Z, h]$. Then call the set $\mathbf{V} = \{\chi_1, \ldots, \chi_n\}$ *independent* (with respect to $\mathscr{A}$) iff

$$\emptyset \neq \bigcap_{i=1}^{n} (P_i \times N_i) \cap h(z, x) \leftrightarrow \bigwedge_{i=1}^{n} (P_i \times N_i \cap h(z, x) \neq \emptyset)$$

for all P_i in ϱ_i, N_i in τ_i, z in Z and x in X.

4.3. Theorem. The NDA $\mathscr{A}$ can be isomorphically embedded into a network of the first kind the output of which is realized by the components iff there exists an independent set $\mathbf{V} = \{\chi_1, \ldots, \chi_n\}$, $n > 1$, of pairs of partitions as defined in 4.2 fulfilling $\prod_{i=1}^{n} \tau_i = 0$ and $\prod_{i=1}^{n} \varrho_i = 0$ where the τ_i's are all nontrivial.

Proof. Analogous to 3.2 if one substitutes f by h and the f_i's by h_i's. □

4.4. Remark. Not every NDA can be embedded isomorphicly into a first kind network according to 4.1.

Proof. This is true even for ND-semiautomata. □

Regarding networks of the second kind, we restrict ourselves again to considering only two-component networks. In the following, let $\chi_1 = [\varrho_1, \tau_1]$ and $\chi_2 = [\varrho_2, \tau_2]$ be pairs of partitions of Y and Z, respectively. We suppose the 'internal' output of component $\mathscr{A}_1$ to be independent of those output signals belonging to the first component of the *network output*. Hence, $Y_1 = Y_1^E \times Y_1^I$, and we can 'decompose' h_1 into two functions, h_1^E and h_1^I in the following way:

$$h_1(z_1, [x, z_2]) =$$
$$= \bigcup_{z'_1 \in f_1(z_1, [x, z_2])} h^E_{1;z'_1}(z_1, [x, z_2]) \times h^I_{1;z'_1}(z_1, [x, z_2]) \times \{z'_1\} .$$

As the second component automaton provides only its present state to be used as "internal information", we identify h_2 and h_2^E (see Fig. 4).

If h_1^E and h_1^I depend on one another, we can regard $\mathscr{A}_1$ to have only one output function h_1.

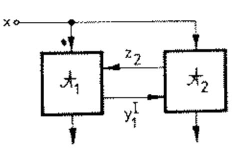


Fig. 4. A second kind network of two components where the network output is realized by the component automata.

4.5. Definition. Let $\mathbf{R}$ be a set system covering f^{τ_1}. χ_2 $\mathbf{R}$-*depends on* χ_1 provided that

$$[y, z'] \in h(z, x) \leftrightarrow (P_1 \times N_1') \cap h(z, x) \neq \emptyset \wedge$$
$$\wedge \exists R(R \in \mathbf{R} \wedge [z, x, N_1'] \in R \wedge \forall \bar{z}(\bar{z} \in N_2 \wedge \{\bar{z}\} \times \{x\} \times \tau_1 \cap R \neq \emptyset \rightarrow$$
$$\rightarrow P_2 \times (u(\bar{z}, R, x) \cap N_2') \cap h(\bar{z}, x) \neq \emptyset))$$

for all P_1 in ϱ_1, P_2 in ϱ_2, y in $P_1 \cap P_2 \neq \emptyset$, N_1, N_1' in τ_1, N_2, N_2' in τ_2, x in X and for some z in $N_1 \cap N_2 \neq \emptyset$ and some z' in $N_1' \cap N_2' \neq \emptyset$.

4.6. Theorem. $\mathscr{A}$ can be embedded into a network of the second kind, the output of which is realized by its components according to 4.1, iff there exist partitions

ϱ_1 and ϱ_2 of Y, nontrivial partitions τ_1 and τ_2 of Z and some $\mathbf{R}$ such that $[\varrho_2, \tau_2]$ $\mathbf{R}$-depends on $[\varrho_1, \tau_1]$ and $\varrho_1 \cdot \varrho_2 = 0_Y$, $\tau_1 \cdot \tau_2 = 0_Z$ (the respective zeropartitions).

Proof. Similar to 3.11 if f is substituted by h. □

5. EXAMPLE

The autonomous NDA $\mathscr{A} = [\{x\}, \{1, 2, 3, 4\}, f]$ is to be decomposed into a network of two components provided f is defined as follows:

f	1	2	3	4
x	$\{1, 3\}$	$\{1, 2\}$	$\{3\}$	$\{1, 2, 4\}$

The only candidates for the state partitions are $\tau_1 = (1, 2/3, 4)$, $\tau_2 = (1, 3/2, 4)$, and $\tau_3 = (1, 4/2, 3)$. However, there does not exist any network of the *first* kind with two components into which $\mathscr{A}$ can be isomorphically embedded because neither $\mathbf{M}_1 = \{\tau_1, \tau_2\}$, $\mathbf{M}_2 = \{\tau_1, \tau_3\}$ nor $\mathbf{M}_3 = \{\tau_2, \tau_3\}$ are independent sets. We select τ_1 and τ_2 for state partitions. Omitting x, we have:

$$f^{\tau_1} = \{1, N_1/1, N_2/2, N_1/3, N_2/4, N_1/4, N_2\}\,.$$

(The blocks of τ_1 and τ_2 are denoted by N_1, N_2 and M_1, M_2, respectively.)

If the second component is in state $\{3, 4\}$, then it needs information about the next state of the first one. That is because $\mathscr{A}$ does not have the next state 3 if the present state is 4, however, it is $f(4, x) \cap \{3, 4\} \neq \emptyset$ and $f(4, x) \cap \{1, 3\} \neq \emptyset$ while $f(4, x) \cap \{3, 4\} \cap \{1, 3\} = \emptyset$. Hence, we must investigate the following cover of f^{τ_1} which separates the transitions $4 \to N_1$ and $4 \to N_2$:

$$\mathbf{R} =_{df} (\underbrace{1, N_1/1, N_2/2, N_1/3, N_2/4, N_1}_{R_1}//\underbrace{1, N_1/1, N_2/2, N_1/3, N_2/4, N_2}_{R_2})\,.$$

This implies:

z	1	2	3	4	1	2	3	4
R	R_1				R_2			
$u(z, R)$	Z	N_1	N_2	N_1	Z	N_1	N_2	N_2

One can easily verify that τ_2 really $\mathbf{R}$-depends on τ_1. Now we construct $\mathscr{A}_1$ and $\mathscr{A}_2$: $\mathscr{A}_1 = [\{x\} \times \tau_2, \mathbf{R}, \tau_1, h_1]$ where h_1 is defined as follows:

z_1 \ z_2	M_1	M_2
N_1	$\{[R_1, N_1], [R_1, N_2], [R_2, N_1], [R_2, N_2]\}$	$\{[R_1, N_1], [R_2, N_1]\}$
N_2	$\{[R_1, N_2], [R_2, N_2]\}$	$\{[R_1, N_1], [R_2, N_2]\}$

$\mathscr{A}_2 = [\{x\} \times \mathbf{R}, \tau_2, f_2]$ where f_2:

$z_2 \backslash x_2$	R_1	R_2
M_1	$\{M_1\}$	$\{M_1\}$
M_2	$\{M_1, M_2\}$	$\{M_2\}$

This implies $\mathscr{A}' = [\{x\}, \tau_1 \times \tau_2, f']$ where

f'	$[N_1, M_1]$	$[N_1, M_2]$	$[N_2, M_1]$	$[N_2, M_2]$
x	$\{[N_1, M_1], [N_2, M_1]\}$	$\{[N_1, M_1], [N_1, M_2]\}$	$\{[N_2, M_1]\}$	$\{[N_1, M_1], [N_1, M_2], [N_2, M_2]\}$

Obviously, $\mathscr{A}$ is isomorphic to $\mathscr{A}'$.

Note that instead of **R** we could have chosen every smaller cover than the above one, e.g. $\mathbf{R}' = (1, N_1/1, N_2/4, N_1//2, N_1/3, N_2/4, N_2)$. The only restriction is that $[4, N_1]$ and $[4, N_2]$ are separated. Possibly, this freedom can be used for finding 'simple' components.

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